Manifold Construction and Parameterization for Nonlinear Manifold-Based Model Reduction

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Outline

- **Background**
  - Introduction to MOR and maniMOR
  - Manifold construction and parameterization

- **Manifold construction using integral curves**
  - DC manifold and the normalized integral curve equation
  - Ideal and almost-ideal manifold
  - Algorithm

- **Experimental results**

- **Conclusion**
Background
Model Order Reduction

Original system (size n)

\[ \frac{d\tilde{x}}{dt} = f(\tilde{x}) + B\tilde{u}(t) \]
\[ \tilde{y} = C\tilde{x} \]

Reduced system (size q)

\[ \frac{d\tilde{z}}{dt} = f_r(\tilde{z}) + B_r\tilde{u}(t) \]
\[ \tilde{y} = C_r\tilde{z} \]

\( \nu: \tilde{x} \mapsto \tilde{z}, \quad \tilde{x} \in \mathbb{R}^n, \tilde{z} \in \mathbb{R}^q, \quad q \ll n \)
Low-order Linear Subspace

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)
\]

Low-order linear subspace

Defined by \( x = Vz \)
Low-order Nonlinear Manifold

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)
\]

Low-order **nonlinear** manifold

**ManiMOR**: MOR Based on Nonlinear Projection on Nonlinear Manifolds
Key Steps in ManiMOR

- **“Find” the nonlinear manifold**
  - Capture important dynamics
- **“Parameterize” the manifold**
  - Build up the coordinate system
Manifold and Its Parameterization

\[
\begin{align*}
  x &= \cos(t) \\
  y &= \sin(t) \\
  z &= t
\end{align*}
\]
**Manifold and Its Parameterization**

**Tangent space** $T_x M \subset \mathbb{R}^n$

**Parameterization of the manifold**

**System of coordinates**

**Manifold defined by pairs of** $\{x, T_x M\}$

No explicit mapping may be derived. Instead, use **piecewise linear approximation**.
Manifold and Its Parameterization

1. Identify the manifold that capture important dynamics

2. Compute and store pairs of $\{x, T_x M\} / \{z, T_z M\}$
DC Manifold

DC operating points constitute a DC manifold.

$$\frac{d\vec{x}}{dt} = f(\vec{x}) + B\vec{u}(t) = 0$$

How to **compute** and **parameterize** the DC manifold?
DC Manifold

\[ f(\vec{x}) + B\vec{u}(t) = 0 \]

**A straight-forward solution:**

**Computation:** Perform DC sweep analysis

**Parameterization:** Define \( \vec{z} \) coordinates using values of \( \vec{u} \)

**Problems:**

Hard to choose step size in DC sweep analysis

Not generalizable to higher dimensions
Introduction to Integral Curve

Given a vector field $v(x)$, its **integral curve** is the curve $\gamma \equiv x(t)$ such that $\frac{dx}{dt} = v(x)$.
DC Manifold as an Integral Curve

Need to derive the relationship between $dx$ and $du$

\[ f(\vec{x}) + B\vec{u}(t) = 0 \]

\[ \frac{\partial f}{\partial x} \frac{dx}{du} + B = 0 \]

\[ \frac{dx}{du} = -[G(x)]^{-1} B \]

Initial condition: \[ x(u = 0) = x_{DC} \mid _{u=0} \]

Solutions are DC operating points.

Any numerical integration / transient analysis code can be applied.

The first Krylov basis.
Parameterization using Euclidean Distance

Parameterization using values of $u$

$$(x_1, y_1) \quad u_0$$
$$(x_2, y_2) \quad u_0 + h$$
$$(x_3, y_3) \quad u_0 + 2h$$

Sample points equally spaced on the DC manifold
Parameterization using Euclidean Distance

Output

Large Error

Input

Output

Better Approximation

Input
Normalized Integral Curve Equation

Local Euclidean distance is

\[ \|dx\|_2 = |du| \]

\[ \left\| \frac{dx}{du} \right\|_2 = \|G(x)^{-1}B\|_2 = 1 \]

Generally not satisfied

Normalize RHS

Normalized Integral Curve Equation

\[ \frac{dx}{du} = -G(x)^{-1}B \]

Does it define the same integral curve?
Validation

state space

- regular integral curve
- normalized integral curve

$x_2$ vs. $x_1$
**Theorem:**

Suppose \( t = \sigma(\tau) \); \( x(t) \) and \( \hat{x}(\tau) \) satisfy

\[
\frac{d}{dt} x(t) = g(x(t)) \quad \text{and} \quad \frac{d}{d\tau} \hat{x}(\tau) = \sigma'(\tau) g(\hat{x}(\tau)),
\]

respectively. Then \( x(t) \) and \( \hat{x}(\tau) \) span the same state space.

**Sketch of proof:**

Since \( t = \sigma(\tau) \), we have \( dt = \sigma'(\tau) d\tau \).

Define \( \hat{x}(\tau) \equiv x(t) = \hat{x}(\sigma(t)) \), then

\[
\frac{d}{d\tau} \hat{x}(\tau) = \frac{d\hat{x}(\tau)}{dt} \frac{dt}{d\tau} = \sigma'(\tau) g(x(t)) = \sigma'(\tau) g(\hat{x}(\tau)).
\]
Normalized Integral Curve Equation

\[
\frac{dx}{du} = -[G(x)]^{-1} B
\]

\[
\frac{dx}{du} = \frac{[G(x)]^{-1} B}{\| [G(x)]^{-1} B \|_2}
\]

Solution: \( x(u) \)  
Solution: \( \hat{x}(\hat{u}) \)

Define

\[
\sigma(\hat{u}) = \int_{0}^{\hat{u}} \frac{1}{\| [G(\hat{x}(\mu))]^{-1} B \|_2} \, d\mu
\]

From the theorem, \( x(u) \) and \( \hat{x}(\hat{u}) \) define the same integral curve.
Normalized Integral Curve Equation

\[ \frac{dx}{du} = \frac{[G(x)]^{-1} B}{\| [G(x)]^{-1} B \|_2} \]

The first normalized Krylov basis.

Directly available from Krylov subspace methods.

Generalizable to higher dimensions.
Ideal Nonlinear Manifold

\[
\begin{align*}
\frac{\partial x}{\partial z_1} &= v_1(x), & \frac{\partial x}{\partial z_2} &= v_2(x), & \cdots, & \frac{\partial x}{\partial z_q} &= v_q(x).
\end{align*}
\]

\[V(x) = [v_1(x), \cdots, v_q(x)]\] is the projection matrix for the reduced linearized system (at \(x\)).

For example, Arnoldi algorithm generates a basis for

\[\mathcal{K}_q([G(x)]^{-1}, B) = \{[G(x)]^{-1}B, [G(x)]^{-2}B, \cdots, [G(x)]^{-q}B\}\]

However, this set of PDEs is over-determined.
Almost-Ideal Manifold Construction

\[ \frac{\partial x}{\partial z_1} = v_1(x) \]

\[ \frac{\partial x}{\partial z_2} = v_2(x) \]

\[ \frac{\partial x}{\partial z_3} = v_3(x) \]
Algorithm 1 Manifold Construction by Finding Integral Curves

1: Given the region to be parameterized \((z_{i,min}, z_{i,max}), i \in [1, q]\);
2: Let \(x_0(0, \cdots, 0) = x_{DC}\), where \(x_{DC}\) is the DC solution when \(u = 0\);
3: \(X \leftarrow \{x_0\}, Z \leftarrow (0, \cdots, 0)\);
4: for \(i = 1\) to \(q\) do
5:    for all \(x \in X\) do
6:        Integrate the integral curve equation
7:        \[\frac{\partial x}{\partial z_i} = v_i(x)\]
8:        with initial condition \(x\);
9:    \(X \leftarrow \{x(z)\}, Z \leftarrow z\);
10: end for
11: end for
12: Output \(X\) as the set of points on the manifold;
13: Output \(Z\) as the parameterization of the manifold for each point \(x \in X\).
Experimental Results
A Hand-Calculable Example

\[
\frac{d}{dt} x_1 = -x_1 + x_2 - u(t) \\
\frac{d}{dt} x_2 = x_1^2 - x_2
\]

\[
f(x) = \begin{bmatrix} -x_1 + x_2 \\ x_1^2 - x_2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\]

\[
G(x) = \begin{bmatrix} -1 & 1 \\ 2x_1 & -1 \end{bmatrix}, \quad [G(x)]^{-1} = \frac{1}{2x_1 - 1} \begin{bmatrix} 1 & 1 \\ 2x_1 & 1 \end{bmatrix}
\]
DC and AC Manifold

\[
[G(x)]^{-1} = \frac{1}{2x_1 - 1} \begin{bmatrix} 1 & 1 \\ 2x_1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\]

\[
w_1(x) = [G(x)]^{-1} B = \frac{1}{2x_1 - 1} \begin{bmatrix} 1 \\ 2x_1 \end{bmatrix}
\]

\[
w_2(x) = [G(x)]^{-2} B = \frac{1}{(2x_1 - 1)^2} \begin{bmatrix} -1 - 2x_1 \\ -4x_1 \end{bmatrix}
\]

DC manifold: \[
\frac{\partial x}{\partial z_1} = v_1(x) = \frac{w_1(x)}{\|w_1(x)\|_2}
\]

AC manifold: \[
\frac{\partial x}{\partial z_1} = v_2(x) = \frac{w_2 - \langle w_2, v_1 \rangle v_1}{\|w_2 - \langle w_2, v_1 \rangle v_1\|_2}
\]
DC and AC Manifold

State space

$x_2$

$x_1$
Application to MOR

\[
\frac{d}{dt} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 
\end{bmatrix} = \begin{bmatrix}
  -10 & 1 & 1 \\
  1 & -1 & 0 \\
  1 & 0 & -1 
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 
\end{bmatrix} - \begin{bmatrix}
  0 \\
  0 \\
  x_1^2 
\end{bmatrix} + \begin{bmatrix}
  1 \\
  0 \\
  0 
\end{bmatrix} u(t)
\]

Trajectory of the full system stays close to the manifold
Simulation of the Reduced Order Model

Response to a step input

Response to a sinusoidal input
Conclusion

- **Presented a manifold construction and parameterization procedure**
  - Based on computing integral curves
  - Preserves local distance
  - Captures important system responses
    - Such as DC and AC responses

- **Application to manifold-based MOR**
  - Validated against several examples