RECAP

3 different types of circuit analysis

DC: all voltages/currents unchanging over time

- simple algebraic equations: \( i = \frac{v}{R} \), \( v_2 = \frac{v_1}{R_1} \)

covered in 16A

Transient: changing with time

- differential equations: \( i(t) = C \frac{dv}{dt} \)  \( \frac{d^2v}{dt^2} + \frac{v(t)}{R} (\alpha \frac{dv}{dt} + \Delta v(t)) \)

- solution: \( v(t) = V_0 e^{-\Delta t} \)

- all you need to know does not change

- special case: everything changing sinusoidally with time

- special type of convenient analysis: sinusoidal S-state

phasors

freq. domain transfer functions

all solving the same LTI problem

in different ways

Transient: most general (but most involved)

DC: easy and important (but limited uses)

sinusoidal steady-state: almost transient (but easier and more insightful than transient)

why are sinusoids important / interesting

demo
Voice/music can be represented as a sum of many sinusoids (at different freqs.).

Changing the relative strengths of different sinusoids: filtering.

Sinusoidal analysis: step towards understanding this.

Later: DFT; later: EE 120.

Last class:

\[ i(t) = \frac{V_0 C \omega \cos(\omega t)}{\sin(\omega t)} \]

\[ i(t) = C \frac{d}{dt} \left( V_0 \sin(\omega t) \right) \]

\[ i(t) = V_0 C \omega \cos(\omega t) \]

Compare with resistor:

\[ i(t) = \frac{V(t)}{R} \]

\[ R = \frac{V(t)}{i(t)} \]

\[ \frac{V_0 \sin(\omega t)}{R \sin(\omega t)} = R \]

Check:
What is \( \frac{v(t)}{i(t)} \) for a cap. with sin. input?

\[
\text{CAP: } \frac{v(t)}{i(t)} = \frac{\text{C} \cdot \sin(\omega t)}{\text{C} \cdot \omega \cos(\omega t)} = \frac{\text{Im}(\omega t)}{\text{Re}}
\]

\( \text{deg} \)???

- "Resistance" depends on time?
- "depends on input freq."
- "goes to 0?"
- "becomes -ve?"

- Seems to complicated compared to res.
- Can we do better?

- The key (as it turns out): express \( \sin() / \cos() \) using complex numbers.

Resources: Tutorial by TA (up last night)
- Short recap: this week's discussion.
- Very short recap:
  - \( j = \sqrt{-1} \): Imaginary number \( \Rightarrow j^2 = -1 \)
  - \( a = a_r + j a_i \): complex number
    \( \uparrow \quad \uparrow \)
    \( \text{real part} \quad \text{imag. part} \)
  - Follows usual rules \( \rightarrow \) addition/multi./etc:
    \[
    c = a + b = (a_r + j a_i) + (b_r + j b_i) = (a_r + b_r) + j (a_i + b_i)
    \]
  - Conjugate of complex numbers:
    Given \( a = a_r + j a_i \)
    \[
    \overline{a} = a_r - j a_i \quad \text{(change sign of imag. part)}
    \]


\[ a + \bar{a} = \text{ALWAYS REAL!} \quad \text{v. IMP.} \]
\[ (av + jai) + (av - jai) = 2av \quad \uparrow \text{Verl} \]

\[ a - \bar{a} = \text{ALWAYS IMAG.} = 2jai \quad \text{EXTREMELY IMP.} \]

\[ \Rightarrow \text{RELATION TO } \sin() / \cos(): \text{ de MOIVRE'S THM.} \]

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]

\[ \Rightarrow \text{USING dMT, can write } \sin() / \cos() \text{ using } e^{j\theta} \] and \[ e^{-j\theta}: \]

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]
\[ + \quad e^{-j\theta} = \cos(\theta) - j\sin(\theta) \]
\[ \frac{e^{j\theta} + e^{-j\theta}}{2} = 2 \cos(\theta) \]

\[ \Rightarrow \cos(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \]

\[ \Rightarrow \text{SUBTRACT: } \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \]

\[ \text{WILL USE THESE} \]

\[ i(t) \rightarrow v_0 \omega \cos(\omega t) \]

\[ V(t) \]
\[ \frac{1}{C} \]
\[ i(t) \rightarrow V(t) \]
\[ u(t) \]
\[ \Rightarrow \quad i(t) = \frac{C}{\omega} \left( v_0 \sin(\omega t) \right) \]

\[ v_0 \sin(\omega t) \]

\[ \Rightarrow v_0 \sin(\omega t) = \frac{v_0}{2j} \left[ e^{j\omega t} - e^{-j\omega t} \right] \Rightarrow i(t) = \frac{C}{\omega} \frac{d}{dt} \left[ e^{j\omega t} - e^{-j\omega t} \right] \]

\[ \Rightarrow i(t) = \frac{v_0}{2j} \frac{Cj\omega}{\omega} \left[ e^{j\omega t} + e^{-j\omega t} \right] = v_0 \omega C \frac{1}{2j} \left[ e^{j\omega t} + e^{-j\omega t} \right] \]

\[ \Rightarrow \cos(\omega t) \]
For the Cap w sin. input:

\[ u(t) = \frac{V_0}{2j} e^{j\omega t} - \frac{V_0}{2j} e^{-j\omega t} \]

\[ i(t) = \frac{V_0}{2} e^{j\omega t} + \frac{V_0}{2} e^{-j\omega t} \]

Look at just the coeffs of \( e^{j\omega t} \):

\[ \tilde{V} = \frac{V_0}{2j}, \quad \tilde{I} = \frac{V_0}{2} C \omega \]

\[ \text{Phasors} \]

\[ u(t) = \tilde{V} e^{j\omega t} + \tilde{I} e^{-j\omega t} \]

\[ i(t) = \frac{\tilde{V}}{j} e^{j\omega t} + \frac{\tilde{I}}{j} e^{-j\omega t} \]

Complex conjugates:

\[ \tilde{V} = \frac{V_0}{2j} \]

\[ \tilde{I} = \frac{V_0}{2} C \omega \]

\[ i(t) = \frac{\tilde{V}}{j} e^{j\omega t} + \frac{\tilde{I}}{j} e^{-j\omega t} \]

Complex conjugates!

\[ \text{So: They always add to real} \]

\[ v(t) / i(t) \text{ always real and \underline{never complex}!} \]

\[ \wedge \text{Never!} \]

But their phasors are complex

Why are phasors useful / interesting?

Try:

\[ \frac{\tilde{V}}{\tilde{I}} = \frac{\frac{V_0}{2j}}{\frac{V_0}{2} C \omega} = \frac{j}{j\omega C} \]

Like a resistance, but depends on \( \omega \)

\[ \text{Angular freq.} \]

\[ \text{"Impedance" of capacitor} \]

\[ \tilde{Z} = \frac{\tilde{V}}{\tilde{I}} \]

\[ \tilde{Z} = \frac{1}{j\omega C} \]

\[ \text{"Impedance" of capacitor} \]
\[ \tilde{X} = \tilde{V} \frac{Z_c}{R + Z_c} = \tilde{V} \left( \frac{1}{R + \frac{1}{j\omega C}} \right) - \tilde{V} \frac{1}{1 + j\omega RC} \]

\[ \tilde{X} = \tilde{V} \left( \frac{1}{1 + j\omega RC} \right) \]

Voltage Transfer Function

\[ H(\omega) \]

Freq. Domain (≡ Phasor Domain)

\[ \rightarrow \text{ENABLES A SYSTEM VIEW OF THE CRT.} \]
WHAT HAVE WE ACHIEVED?

GIVEN Sinusoidal \( v(t) = \sqrt{V} e^{j\omega t} + \text{c.c. term} \)

FOUND \( \tilde{X} = \text{phasor of } v(t) = \sqrt{V} H(\omega) = \frac{\sqrt{V}}{1 + j\omega RC} \)

Now we can find \( x(t) \) very easily.

\[ x(t) = \sqrt{V} e^{j\omega t} + \text{c.c. term} \]

Express in polar form: \( |\tilde{X}| e^{j\theta} \)

DISCUSSION

\[ |\tilde{X}(\omega)| = \frac{|\tilde{U}|}{\sqrt{1 + \omega^2 RC^2}}, \quad \theta(\omega) = \angle H(\omega) = -\tan^{-1}(\omega RC) \]

\[ x(t) = |\tilde{X}(\omega)| e^{j(\omega t + \theta)} + \text{c.c. term} \]

\[ x(t) = 2 |\tilde{X}(\omega)| \cos(\omega t + \theta) \]

Oscilloscope DEMO (numerical)
OSCILLOSCOPE DEMO

**Summary:** What have we done today

- **Demo (Audio):** Many sinusoids at different freqs = voice/music/etc.
- Sinusoids can be written using complex numbers + c.c.
  - **Phasor:** coeff. of \( e^{j\omega t} \)
  - In "Phasor Domain": Cap looks like resistor
    - Impedance depends on freq \( \omega \)
- Circuits can be solved in Phasor Domain = Z(s)
  - Big win over solving diff. eqns.
  - But valid only if all waveforms sinusoidal.

- In Phasor Domain: ratios of voltages/currents are simple expressions in \( s \) (e.g., \( \frac{1}{1 + js \omega C} \))

  - **Transfer Functions** \( H(s) \): output = \( H(s) \) \* input phasors
    - "Freq. Domain"

- Can easily recover time domain (e.g. \( v(t) \), \( i(t) \)) from phasors.