THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

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\[ N = \text{odd} = 2M + 1 \]

\[ \downarrow \]

\[ \frac{1}{N} F_N^* \]

**TIME DOMAIN**

\[ \hat{X} \]

**DC SIGNAL**

\[ \rightarrow x(t) = A_0 \quad \leftarrow \text{DC REAL} \]

\[ \rightarrow x(t) = x[k] = a_k \quad (k = 0, \ldots, N-1) \]

**FREQUENCY DOMAIN**

\[ X \]

\[ \rightarrow X_0 = N A_0 \quad \leftarrow \text{REAL} \]

\[ \rightarrow X_1, X_2, \ldots, X_N = 0 \]

**SINUSOID AT FUNDAMENTAL:** FREQUENCY = \( \frac{2\pi}{N} \)

\[ x(t) = A_1 \cos \left( \frac{2\pi}{N} t + \phi \right) \quad \leftarrow \text{REAL} \]

\[ \rightarrow \text{phasor} A_1 e^{\phi} \quad \text{of} \quad \omega = \frac{2\pi}{N} \]

\[ \rightarrow x_k = x(k), \quad (k = 0, \ldots, N-1) \]

**SINUSOID AT 2^{th} HARMONIC:** FREQUENCY = \( 2 \times \frac{2\pi}{N} \)

\[ x(t) = A_2 \cos \left( \frac{2\pi}{N} 2t + \phi \right) \quad \leftarrow \text{REAL} \]

\[ \rightarrow \text{phasor} A_2 e^{\phi} \quad \text{of} \quad \omega = \frac{2\pi}{N} \]

\[ \rightarrow x_k = x(k), \quad (k = 0, \ldots, N-1) \]
**SINUSOID AT 3rd HARMONIC: FREQUENCY = \(3 \times \frac{2\pi}{N}\)**

- \(x(t) = A_3 \cos(3 \frac{2\pi}{N} t + \phi_3) \) \(\rightarrow\) REAL
- \(\text{phasor: } \frac{A_3}{2} e^{j\phi_3}, \text{ af } w = 3 \frac{2\pi}{N}\)
- \(x_k = x(t), \ t = 0, \ldots, N-1\)

**COMPLEX**
- \(X_3 = N \frac{A_3}{2} e^{j\phi_3}\)
- \(X_{-3} = X_3 = \bar{X}_3\)
- \(X_0, X_1, X_2, X_4, X_5, X_6, X_7, X_8, X_9 = 0\)

**SINUSOID AT 5th HARMONIC: FREQUENCY = \(5 \times \frac{2\pi}{N}\)**

- \(x(t) = A_5 \cos(5 \frac{2\pi}{N} t + \phi_5) \) \(\rightarrow\) REAL
- \(\text{phasor: } \frac{A_5}{2} e^{j\phi_5}, \text{ af } w = 5 \frac{2\pi}{N}\)
- \(x_k = x(t), \ t = 0, \ldots, N-1\)

**COMPLEX**
- \(X_5 = N \frac{A_5}{2} e^{j\phi_5}\)
- \(X_{-5} = X_5 = \bar{X}_5\)
- \(X_0, X_1, X_2, \ldots, X_{4,5}, X_{5,1}, X_{5,2}, \ldots, X_{N-1} = 0\)

**SUM OF ALL THESE SINUSOIDS AT HARMONICALLY RELATED FREQUENCIES (+ DC)**

- \(x(t) = A_0 + \sum_{i=1}^{M} A_i \cos \left( \frac{2\pi}{N} t + \phi_i \right) \) \(\rightarrow\) REAL
- \(x_k = x(t), \ t = 0, \ldots, N-1\)

**REAL**
- \(X = F_N x\)

**GENERATED IN MATLAB**

- Harmonic 0: A=1, \(\phi=0\)
- Harmonic 1: A=2, \(\phi=0\)
- Harmonic 2: A=1.5, \(\phi=30\)
- Harmonic 3: A=1, \(\phi=60\)
- Harmonic 4: A=0.5, \(\phi=90\)
- Harmonic 5: A=0.25, \(\phi=120\)

**COHANGATES**
- \(A_{5,1} = \frac{\sqrt{3}}{2}\)
- \(A_{5,2} = \sqrt{2}\)
- \(A_{5,3} = \frac{1}{2}\)
- \(A_{5,4} = 1\)
- \(A_{5,5} = 0\)
- \(A_{5,6} = -1\times N\)
KEY POINT

- TAKE ANY ODD $N > 1$, $N = 2M + 1$

- SOMEONE GIVES YOU $N$ SAMPLES OF $x(t) = a_0 + \sum_{i=1}^{N} a_i \cos \left( \frac{2\pi t}{N} \theta_i \right)$, $t = 0, 1, \ldots, N-1$

- WITHOUT TELLING YOU $a_0, a_i, \theta_i$ — i.e., YOU DON'T KNOW THE PHASORS $A_i e^{j\theta_i}$, OR $A_0$

- JUST TAKE A DFT: $X = F_n \mathbb{R}$

- $\frac{1}{N} X_i$ ARE YOUR PHASORS ($i = 1, \ldots, M$); $\frac{1}{N} X_0$ IS THE DC TERM!

- ALL YOU NEED ARE THE SAMPLES; THE DFT GETS YOU THE PHASORS IMMEDIATELY

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CHANGING THE TIME PERIOD / FUNDAMENTAL FREQUENCY

- We had $x(t) = a_0 + \sum_{i=1}^{M} a_i \cos \left( \frac{2\pi t}{N} \theta_i \right)$

- THE TIME PERIOD IS $N$ — RECALL: $\cos \left( \frac{2\pi t}{T} \right)$ has a period $T$

- FUNDAMENTAL FREQUENCY = $\frac{1}{\text{TIME PERIOD}} = \frac{1}{N}$

- $x(t) = a_0 + \sum_{f=1}^{M} A_f \cos \left( \frac{2\pi}{N} \frac{f}{f} t + \theta_i \right)$

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Diagram:
- Harmonics 0, 1, 2, 3, 4, 5, 6
- $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$
- $\theta_0$, $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $\theta_5$, $\theta_6$

- Time Period = $N$

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Conjugates:
- Phasors
- Fundamental
- 2nd harmonic
- 4th harmonic

- $f_0$, $f_1$, $f_2$, $f_3$, $f_4$
Suppose we want a different fundamental frequency

\[ x(t) = A_0 + \sum_{i=1}^{M} A_f \cos \left( 2\pi f_0 i t + \theta_i \right) \]

\[ \Rightarrow \text{Time period} \quad T = \frac{1}{f_0} \]

I still take \( N = 2M + 1 \) samples, spaced at \( A = \frac{T}{N} \)

\[ x = \begin{bmatrix} x(0) \\ x(2A) \\ \vdots \\ x((N-1)A) \end{bmatrix} \]

\[ \hat{x} = \begin{bmatrix} \hat{x}_0 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{N-1} \end{bmatrix} \]

\[ \hat{x}(n) = \sum_{k=0}^{N-1} x(k) e^{-j2\pi nk/N} \]

\[ x(n) = \sum_{k=0}^{N-1} \hat{x}(k) e^{j2\pi nk/N} \]

Is the DFT of \( \hat{x} \) still useful?

Yes: it still gives you the phasors.

The fundamental and harmonic frequencies are different:

\( f_0, 2f_0, \ldots, Mf_0 \)

\[ \Rightarrow \text{instead of} \quad \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N} \]

Demo: Calling Elvis in MATLAB

CD Format: Sample rate = 44,100 samples per second

\[ A = \frac{1}{44,100} \Rightarrow \text{Spacing of TD samples} \]

We'll take 1 minute = 60s of audio data

\[ = 60 \times 44,100 = 2,646,000 \text{ samples} \]

We'll take \( N = 2,646,001 \) (to make it odd)

\[ M = \frac{N-1}{2} = 1,322,500 \]
\[ T = \frac{2,666,001}{44,100} = 60 + 2.267 \times 10^5 \text{ s} \]

\[ \text{FUNDAMENTAL FREQUENCY} \quad f_0 = \frac{1}{T} = 1.667 \times 10^2 \text{ Hz} \]

\[ \text{HIGHEST HARMONIC} \quad Mf_0 = \frac{M}{N} = \frac{M}{N} \times 44,100 = \frac{M}{2N+1} \times 44,100 \approx 25,075 \text{ Hz} \]

\[ \text{DFT SIZE: } N \times N \]

\[ F_n \bar{X} : \text{HOW MANY MULTIPLICATION OPERATIONS?} \]

\[ \text{MY COMPUTER'S CLOCK PERIOD: } \frac{1}{4 \text{GHz}} = 0.25 \times 10^{-9} \text{ s} \]

\[ \text{MINIMUM TIME NEEDED FOR ANY OPERATION} \]

\[ \text{COMPUTER TIME NEEDED FOR } F_n \bar{X} : \quad x \times 0.25 \times 10^{-9} \]

\[ \text{RUN MATLAB CODE } \text{callingelvis.m}. \]

\[ \frac{X_i}{2} = A_i e^{i0_i} \]

\[ X(f), f \approx f_0 \quad i = 0, \ldots, N \]

\[ \text{FREQUENCY SPECTRUM OF THE AUDIO SIGNAL} \]

\[ \text{RECALL (?) THE INVERSE DFT (IDFT)} \]

\[ \check{X} = F_N \tilde{X} \iff \tilde{X} = F_N^{-1} \check{X} \]

\[ \text{CAN SHOW (see notes) that } F_N = \frac{1}{N} F_N^* \]
INTERPOLATION VIA THE DFT

\[ x(\theta) = A_0 + \sum_{i=1}^{N_e} A_i \cos (2\pi f_0 i \theta + \phi_i) \]

TD

N = 11 TD SAMPLES

FD

N = 17 TD POINTS

MORE SAMPLES VIA "BAND LIMITED" INTERPOLATION

INCREASE M IN THE FD REPRESENTATION:

- \[ M = 8 \Rightarrow N = 2M + 1 = 17 \]

DON'T CHANGE THE SPECTRUM : JUST PAD WITH ZEROS

F_0 (FUNDAMENTAL) REMAINS THE SAME

SPECTRUM REMAINS THE SAME

- \[ x(\theta) = A_0 + \sum_{i=1}^{N_e} A_i \cos (2\pi f_0 i \theta + \phi_i) \]

- \[ A_0 = A_2 = A_4 = 0 : \text{NO CHANGE TO} x(\theta) \]

DFT-BASED (BAND-LIMITED) INTERPOLATION : THE PROCEDURE

- START WITH \( \mathbf{X} \in \mathbb{C}^N \), \( N = 2M + 1 \) \( \text{representing samples at} \ 0, \frac{T}{N}, \frac{2T}{N}, ..., \frac{(N-1)T}{N} \)

- \( \mathbf{X} = F_N \mathbf{X} \) \( \text{SIZE} \ N \ \text{DFT} \)

- CHOOSE SOME \( M > M \) : DEFINE \( N = 2M + 1 \)

- DEFINE \( \mathbf{X}_2 \) BY:

- \( \mathbf{X}_2[0, \ldots, M] = \frac{N_2}{N} \mathbf{X}[0, \ldots, M] \) \( \text{COPY THE PHASORS OF THE ORIGINAL HARMONICS} \)

- \( \mathbf{X}_2[M+1, \ldots, N-1] = \frac{N_2}{N} \mathbf{X}[M+1, \ldots, N-1] \) \( \text{COPY THE CONJUGATES TO THE RIGHT PLACES} \)

- \( \mathbf{X}_2[M+1, \ldots, N-1-M] = 0 \) \( \text{PAD THE EXTRA HARMONICS TO ZEROS} \)

- \( \mathbf{X}_2 = F_{N_2}^{\dagger} \mathbf{X}_2 \) \( \text{SIZE} \ N_2 \ \text{INVERSE DFT} \)

INTERPOLATED SAMPLES \( \mathbf{X}_2[0, \frac{T}{N_2}, \frac{2T}{N_2}, \ldots, \frac{(N_2-1)T}{N_2}] \)