THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES
2. ROOTS OF 1 ON THE COMPLEX PLANE
3. THE DFT MATRIX
4. VECTORS FROM SEQUENCES
5. FREQUENCY INTERPRETATION OF DFT SIGNALS — WITHOUT PROOFS
6. DEMO: MATLAB
7. INTERPOLATION VIA THE DFT
DFT: \( \hat{X} = F_N X \)

- \( N = \text{odd} = 2N + 1 \)
- \( \downarrow \)
- \( \hat{X} = \sum_{k=0}^{N-1} x_k \)

**Time Domain**

- \( x(t) : \text{DC} \)  
- \( x(t) = a_n, \; t = 0, \ldots, N-1 \)

**Frequency Domain**

- \( X_0 = N a_0 \)  
- \( X_1, X_2, \ldots, X_N = 0 \)

**Sinusoid at Fundamental Frequency:**

- \( X(f) = A_0 \cos \left( \frac{2\pi}{N} f \right) \)  
- \( N \) is an integer

**Sinusoid at 2\(^{th}\) Harmonic Frequency:**

- \( X(f) = A_2 \cos \left( \frac{2\pi}{N} (f + \frac{N}{2}) \right) \)  
- \( N \) is an integer

- \( X_1 = \frac{A_2}{2} e^{i\theta_2} \)  
- \( X_2 = \frac{A_2}{2} e^{i\theta_2} \)  
- \( X_{N-2} = X_N = \bar{X}_2 \)  
- \( X_0, X_1, X_2, \ldots, X_N, X_{N-1} = 0 \)
\[ A_1 \cos(2\pi ft + \theta_1) = \frac{A_1}{2} \left( e^{j\theta_1} e^{j2\pi ft} + e^{-j\theta_1} e^{-j2\pi ft} \right) \]

\[ \Rightarrow \frac{A_1}{2} \left( e^{j\theta_1} e^{j2\pi ft} \right) = e^{j2\pi ft} \frac{A_1}{2} e^{j\theta_1} \]

\[ \Rightarrow 2 \cos(2\pi ft + \theta_1) = e^{j\theta_1} e^{j2\pi ft} \]

\[ + j \sin(2\pi ft + \theta_1) \]

\[ (B e^{j\theta} + B^{-j\theta}) \]
**SINUSOID AT 3rd HARMONIC:** Frequency $= 3 \times \frac{2\pi}{N}$

- $x(t) = A_3 \cos(3 \times \frac{2\pi}{N} t + \theta_3)$ ← REAL
- $z[k] = x[k], \quad k = 0, \ldots, N-1$

**SINUSOID AT Mth HARMONIC:** Frequency $= M \times \frac{2\pi}{N}$

- $x(t) = A_M \cos(M \times \frac{2\pi}{N} t + \theta_M)$ ← REAL
- $z[k] = x[k], \quad k = 0, \ldots, N-1$

**SUM OF ALL THESE SINUSOIDS AT HARMONICALLY RELATED FREQUENCIES (+ DC)**

- $x(t) = \sum_{i=0}^{M} A_i \cos\left(\frac{2\pi}{N} i t + \theta_i\right)$ ← REAL
- $z[k] = x[k], \quad k = 0, \ldots, N-1$

**GENERATED IN MATLAB**

- $x(t)$ and $z[k]$ graphs are shown for different harmonics.

**FINDING THE FD OF EACH HARMONIC**

- $X_M = A_M e^{j\theta_M}$ ← COMPLEX
- $X_{N-M} = X_M^*$
- $X_{0}, X_1, X_2, \ldots, X_{N-1} = 0$

**CONVERTING 2D TO 3D**

- $X = F_N x$
- $F_N$ transform is shown in 3D space.
→ **KEY POINT**

→ **TAKE ANY ODD** \( N > 1 \), \( N = 2M + 1 \)

→ **SOMEONE GIVES YOU** \( N \) **SAMPLES OF** \( x(t) = A_0 + \sum_{i=1}^{N} A_i \cos \left( \frac{2\pi i t + \theta_i}{N} \right) \) \( a.t. t = 0, 1, ..., N-1 \)

→ **WITHOUT TELLING YOU** \( A_0, A_i, \theta_i \) \( \Rightarrow \) **YOU DON'T KNOW THE PHASORS** \( \frac{A_i e^{j\theta_i}}{2} \), **OR** \( A_0 \)

→ **JUST TAKE A DFT** : \( X_k = F_N X \)

→ \( \frac{1}{N} \) \( X_i \) **ARE YOUR PHASORS** \( (i=1, \ldots, M) \); \( \frac{1}{N} X_0 \) **IS THE DC TERM**!

→ **ALL YOU NEED ARE THE SAMPLES**; **THE DFT GETS YOU THE PHASORS IMMEDIATELY**

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→ **CHANGING THE TIME PERIOD / FUNDAMENTAL FREQUENCY**

→ **we had** \( x(t) = A_0 + \sum_{i=1}^{N} A_i \cos \left( \frac{2\pi i t + \theta_i}{N} \right) \)

→ **THE TIME PERIOD** is \( N \) \( \Rightarrow \) **RECALL** : \( \cos \left( \frac{2\pi i t}{T} \right) \) has a period \( 2T \).

→ **FUNDAMENTAL FREQ** = \( \frac{1}{\text{TIME PERIOD}} = \frac{1}{N} = f_0 \)

\[ x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( 2\pi \frac{n}{N} t + \theta_n \right) \]

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Suppose we want a different fundamental frequency
\[ x(t) = A_0 + \sum_{i=1}^{M} A_i \cos(2\pi f_i t + \theta_i) \]

\[ T = \frac{1}{f_0} \]

I still take \( N = 2M + 1 \) samples, spaced at \( A = T = \frac{1}{f_0} \)

Is the DFT of \( \hat{x} \) still useful?

Yes: it still gives you the phasors. The fundamental and harmonic frequencies are different

\( f_0, 2f_0, \ldots, Mf_0 \)

Instead of \( \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N} \)

Demo: Calling Elvis in MATLAB

CD format: Sample rate = 44,100 samples per second

\[ A = \frac{1}{44,100} \]

Spacing of TD samples

We'll take 1 minute = 60s of audio data

= 60 x 44,100 = 2,646,000 samples

We'll take \( N = 2,646,001 \) (to make it odd) \( N = 2m + 1 \)

\[ M = \frac{N-1}{2} = 1,323,000 \]
\[ T = \text{N} \times \Delta \Rightarrow T = \frac{2,646,003}{44,100} = 60 + 2.267 \times 10^5 \approx 60 \text{s}. \]

\[ f_0 = \frac{1}{T} = 1.667 \times 10^2 \text{ Hz.} \approx \frac{1}{60} \]

\[ \text{HIGHEST HARMONIC} = Mf_0 = \frac{M}{T} = \frac{M}{N \Delta} = \frac{M}{N} \times 44,100 = \frac{M}{24 \times 6} \times 44,100 \approx 22.05 \text{ Hz} \]

**DFT SIZE:** \[ N \times N \]

\[ \frac{1}{4 \times 10^9} \text{ s} \]

**MY COMPUTER'S CLOCK PERIOD:** \[ n \times \frac{1}{4 \text{ GHz}} = 0.25 \times 10^{-9} \text{ s} \]

**MINIMUM TIME NEEDED FOR ANY OPERATION**

**COMPUTER TIME NEEDED FOR F_N X:** \[ 6 \times 10^{-8} \times 0.25 \times 10^9 = 1.5 \times 10^2 \text{ s} \]

**RUN MATLAB CODE** callingelvis.m.

\[ X = \frac{A_i e^{j0i}}{2} \]

\[ X(f_i, f_0, i=0, \ldots, N) \]

**FREQUENCY SPECTRUM OF THE AUDIO SIGNAL**

**RECALL (I)** **THE INVERSE DFT (IDFT)**

\[ X = F_N x \iff \hat{x} = F_N^\dagger x \]

**CAN SHOW (SEE NOTES)** that \[ F_N = \frac{1}{N} F_N^\dagger \]
**INTERPOLATION VIA THE DFT**

\[ \hat{x}(k) = A_0 + \sum_{i=1}^{M} A_i \cos \left( 2\pi \frac{f_i}{f_0} k + \theta_i \right) \]

- **TD**
  - \( N = 11 \) TD SAMPLES

- **FD**
  - \( N_2 = 17 \) TD POINTS
  - MORE SAMPLES VIA "BAND LIMITED" INTERPOLATION

→ INCREASE M IN THE F.D. REPRESENTATION:
  - \( M_2 = (\text{say}) 8 \Rightarrow N_2 = 2M_2 + 1 = 17 \)

→ DON'T CHANGE THE SPECTRUM: JUST PAD WITH ZEROS

→ \( f_0 \) (FUNDAMENTAL) REMAINS THE SAME
  - SPECTRUM REMAINS THE SAME

→ \( \hat{x}(k) = A_0 + \sum_{i=1}^{M_2} A_{2i} \cos \left( 2\pi \frac{f_i}{f_0} k + \theta_i \right) \)

  - with \( A_0 = A_2 = A_4 = 0 \):
    - NO CHANGE TO \( x(t) \)

→ DFT-BASED (BAND-LIMITED) INTERPOLATION: THE PROCEDURE

→ START WITH \( \overline{x} \in \mathbb{R}^N \), \( N = 2M_2 + 1 \) — representing samples at \( 0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N} \)

→ \( \overline{x} = F_N \overline{x} \) (SIZE \( N \) DFT)

→ CHOOSE SOME \( M_2 > M \) DEFINE \( N_2 = 2M_2 + 1 \)

→ DEFINE \( \hat{x}_2 \) BY:
  - \( \hat{x}_2[0, \ldots, M] = \frac{N_2}{N} \hat{x}[0, \ldots, M] \) — COPY THE PHASORS OF THE ORIGINAL HARMONICS
  - \( \hat{x}_2[N_2-M, \ldots, N_2-1] = \frac{N_2}{N} \hat{x}[M+1, \ldots, N-1] \) — COPY THE CONJUGATES TO THE RIGHT PLACES
  - \( \hat{x}_2[M+1, \ldots, N_2-M-1] = 0 \) — PAD THE EXTRA HARMONICS TO ZEROS.

→ \( \overline{x}_2 = F_N^{-1} \hat{x}_2 \) (SIZE \( N_2 \) INVERSE DFT)

→ INTERPOLATED SAMPLES \( \overline{x} \in \{0, \frac{T}{N_2}, \frac{2T}{N_2}, \ldots, \frac{(N_2-1)T}{N_2}\} \)
\[X(t) = A_0 + \]
\[A_1 \cos(2\pi f_1 t + \theta_1) + \]
\[A_2 \cos(2\pi f_2 t + \theta_2) + \]
\[\vdots \]
\[A_5 \cos(2\pi 5f_1 t + \theta_5) + \]
\[A_6 \cos(2\pi 6f_1 t + \theta_6) + \]
\[A_7 \cos(2\pi 7f_1 t + \theta_7) + \]
\[A_8 \cos(2\pi 8f_1 t + \theta_8) + \]