THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES ✓
2. ROOTS OF 1 ON THE COMPLEX PLANE ✓
3. THE DFT MATRIX ✓
4. VECTORS FROM SEQUENCES ✓
5. FREQUENCY INTERPRETATION OF DFT SIGNALS — WITHOUT PROOFS
6. DEMO: MATLAB
7. INTERPOLATION VIA THE DFT
5. **DFT:** \( \hat{X} = F_N \tilde{x} \)

\[
N = \text{odd} = 2M + 1
\]

\[
\downarrow \quad \text{IDFT} = \frac{1}{N} F_N^* \]

**TIME DOMAIN**

\( \tilde{x} \)

**FREQUENCY DOMAIN**

\( \hat{x} \)

---

**TIME DOMAIN**

\( x(t) \)

**DC SIGNAL**

\( x(t) = x[0] \)

\( x[0] = x_0 \)

**REAL**

\( x(t) = x[k] = a_k, \quad k = 0, \ldots, N-1 \)

**FREQUENCY DOMAIN**

\( X \)

**REAL**

\( X_0 = N \cdot a_0 \)

\( X_1, X_2, \ldots, X_N = 0 \)

---

**TIME DOMAIN**

Sinusoid at fundamental frequency: \( \frac{2\pi}{N} \)

\( x(t) = A_1 \cos\left(\frac{2\pi}{N} t + \theta_1\right) \)

**REAL**

\( x(t) = x[k] = a_k, \quad k = 0, \ldots, N-1 \)

**FREQUENCY DOMAIN**

\( X \)

**COMPLEX**

\( X_1 = N \cdot A_1 e^{i\theta_1} \)

\( X_0, X_2, \ldots, X_N = 0 \)

\( X_0 = x_0 \)

---

**TIME DOMAIN**

Sinusoid at \( 2^m \)th harmonic: \( \frac{2\pi}{N} \)

\( x(t) = A_2 \cos\left(\frac{2\pi}{N} t + \theta_2\right) \)

**REAL**

\( x(t) = x[k] = a_k, \quad k = 0, \ldots, N-1 \)

**FREQUENCY DOMAIN**

\( X \)

**COMPLEX**

\( X_2 = A_2 e^{i\theta_2} \)

\( X_0, X_1, X_3, \ldots, X_N = 0 \)

\( X_0 = x_0 \)

---

\( \hat{x} \)

**DC**

\( x(t) = x[0] \)

\( x_0 = A_0 \)

\( x[0], x[1], \ldots, x[N-1] \)

SINUSOID AT 3rd HARMONIC: \( \text{FREQUENCY} = \frac{3 \times 2\pi}{N} \)

\[ x(k) = A_3 \cos(\frac{3 \times 2\pi k}{N} + \theta_3) \] \( \rightarrow \) \text{REAL}

\[ x_k = x(k), \quad k = 0, \ldots, N-1 \]

\[ X_k = A_3 \frac{e^{j\theta_3}}{2} \] \( \rightarrow \) \text{COMPLEX}

\[ X_0, X_1, X_2, \ldots, X_{N-1} = 0 \]

\[ X_3 = A_3 \frac{e^{j\theta_3}}{2} \]

\[ \text{SUM OF ALL THESE SINUSOIDS AT HARMONICALLY RELATED FREQUENCIES (+ DC)} \]

\[ x(k) = A_0 + \sum_{i=1}^{N} A_i \cos\left(\frac{2\pi i k}{N} + \theta_i\right) \] \( \rightarrow \) \text{REAL}

\[ x_k = x(k), \quad k = 0, \ldots, N-1 \]

\[ X = F_N x \]

\[ X_0 = A_0 \frac{e^{j\theta_0}}{N} \]

\[ A_{11} = \frac{-1}{N} \]

\[ A_{12} = \frac{1}{N} \]

\[ A_{13} = \frac{1}{N} \]

\[ A_{14} = \frac{-1}{N} \]

\[ X_0, X_1, X_2, \ldots, X_{N-1} \] generated in MATLAB.
\[ x(t) = A_0 + \sum_{i=1}^{M} A_i \cos \left( \frac{2\pi i}{N} t + \phi_i \right) \]

**Key Point:**
- Take any odd \( N > 1 \), \( N = 2M + 1 \)
- Someone gives you \( N \) samples of \( x(t) = A_0 + \sum_{i=1}^{N} A_i \cos \left( \frac{2\pi i}{N} t + \phi_i \right) \) for \( t = 0, 1, \ldots, N-1 \)
- Without telling you \( A_0, A_i, \phi_i \), i.e., you don't know the phasors \( A_i e^{i\phi_i} \) or \( A_0 \)
- Just take a DFT: \( X = F_X \mathbf{x} \)
- \( \frac{1}{N} X_i \) are your phasors \((i=1, \ldots, M)\); \( \frac{1}{N} X_0 \) is the DC term!
- All you need are the samples; the DFT gets you the phasors immediately

---

**Changing the Time Period / Fundamental Frequency**
- We had: \( x(t) = A_0 + \sum_{i=1}^{M} A_i \cos \left( \frac{2\pi i}{N} t + \phi_i \right) \)
- The time period is \( N \) (Recall: \( \cos \left( \frac{2\pi t}{T} \right) \) has a period \( 2T \)).
- Fundamental freq = \( \frac{1}{\text{Time Period}} = \frac{1}{N} \)

**Sinc Function**
- \( x(t) = A_0 + \sum_{f=1}^{M} A_f \cos \left( 2\pi \frac{f}{N} t + \phi_i \right) \)

**Diagram:**
- Time Period = \( N \)
Suppose we want a different fundamental frequency

\[ x(t) = A_0 + \sum_{i=1}^{M} A_i \cos(2\pi f_i t + \theta_i) \]

\[ \Rightarrow \text{time-period} \quad T = \frac{1}{f_0} \]

We still take \( N = 2M+1 \) samples, spaced at \( \Delta T = \frac{T}{N} \)

Is the DFT of \( \mathbf{x} \) still useful?

Yes: it still gives you the phasors

The fundamental and harmonic frequencies are different

\( f_0, 2f_0, \ldots, Mf_0 \)

Instead of \( \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N} \)

Demo: Calling Elvis in MATLAB

CD format: Sample rate = 44,100 samples per second

\[ \Delta T = \frac{1}{44,100} \]

Spacing of TD samples

We'll take 1 minute = 60s of audio data

\[ 60 \times 44,100 = 2,646,000 \] samples

We'll take \( N = 2,646,001 \) (to make it odd)

\[ M = \frac{N-1}{2} = 1,323,000 \]
\[ T = \frac{N \cdot \Delta}{44,100} = \frac{2,646,600}{44,100} = 60 + 2.267 \times 10^5 \approx 605 \text{ s}. \]

\[ \text{FUNDAMENTAL FREQ } f_0 = \frac{1}{T} = 1.667 \times 10^2 \text{ Hz}. \]

\[ \text{HIGHEST HARMONIC } = M f_0 = \frac{M}{T} = \frac{M}{N \Delta} = \frac{M}{N} \times 44,100 = \frac{M}{2 \Delta} \times 44,100 \approx 22,050 \text{ Hz}. \]

\[ \text{DFT SIZE: } N \times N \]

\[ F_n \Rightarrow \text{how many multiplication operations?} \]

\[ \text{MY COMPUTER'S CLOCK PERIOD: } \frac{T}{4 \text{GHz}} = 0.25 \times 10^{-9} \Rightarrow \text{needed for any operation} \]

\[ \text{COMPUTER TIME NEEDED FOR } F_n \Rightarrow \text{?} \times 0.25 \times 10^{-9} \]

\[ \text{RUN MATLAB CODE callingelvis.m} \]

\[ \frac{X}{z} = A_i e^{\frac{j \theta_i}{2}} \]

\[ X(f), f = i f_0 \]

\[ i = 0, \ldots, M \]

\[ \text{FREQUENCY SPECTRUM OF THE AUDIO SIGNAL} \]

\[ \text{RECALL (??) THE INVERSE DFT (IDFT)} \]

\[ \tilde{X} = F_N \tilde{X} \Leftrightarrow \tilde{X} = F_N^{-1} \tilde{X} \]

\[ \text{CAN SHOW (see notes) that } F_N = \frac{1}{N} F_N^* \]
INTERPOLATION VIA THE DFT

\[ x(t) = A_0 + \sum_{i=1}^{M} A_i \cos \left( 2\pi f_0 i t + \theta_i \right) \]

**TD**

- **N = 11 TD SAMPLES**

**FD**

- **INCREASE M IN THE F.D. REPRESENTATION:**
  - \( M_0 = (\text{say}) 8 \Rightarrow N_0 = 2M_0 + 1 = 17 \)
  - **DON'T CHANGE THE SPECTRUM:** JUST PAD WITH ZEROS
  - \( N_2 = 17 \text{ TD POINTS} \)

**MORE SAMPLES VIA**

- **"BAND LIMITED" INTERPOLATION**
  - **MORE SAMPLES**

**DFT-BASED (BAND-LIMITED) INTERPOLATION: THE PROCEDURE**

- **START WITH** \( \tilde{x} \in \mathbb{R}^M \), \( N = 2M + 1 \) representing samples at \( 0, \frac{T}{N}, \frac{2T}{N}, ... \)
- \( \tilde{X} = F_N \tilde{x} \) (SIZE N DFT)
- **CHOOSE SOME** \( M > M_0 \) DEFINE \( N_0 = 2M_0 + 1 \)
- **DEFINE** \( \tilde{X}_2 \) BY:
  - \( \tilde{X}_2[0, \ldots, M] = \frac{N_2}{N} \tilde{X}[0, \ldots, M] \) **COPY THE PHASORS OF THE ORIGINAL HARMONICS**
  - \( \tilde{X}_2[N_0-M, \ldots, N_0-1] = \frac{N_2}{N} \tilde{X}[N_0, \ldots, N_0-1] \) **COPY THE CONJUGATES TO THE RIGHT PLACES**
  - \( \tilde{X}_2[N_0, \ldots, N_0-M-1] = 0 \) **PAD THE EXTRA HARMONICS TO ZEROS**
- \( \tilde{x}_2 = F_N^{-1} \tilde{X}_2 \) (SIZE \( N_2 \) INVERSE DFT)
- **INTERPOLATED SAMPLES** at \( 0, \frac{T}{N_2}, \frac{2T}{N_2}, ... \) \( \frac{(N_2-1)T}{N_2} \)