1. DFTs: Introduction

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The DFT is just a matrix (square matrix)

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But a very special matrix

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Filled with (special) complex numbers

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Invertible (and its inverse is "almost itself")

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Widely useful

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Convert "time-domain" to "frequency domain" and vice-versa

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Digital signal processing

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Find frequency spectrum

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Perform filtering

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Fourier analysis / spectrograms / etc.

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Image compression

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JPEG images (DCT)

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Fast computation: FFT

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Show live demo + slides

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Can be used for a special kind of interpolation (band-limited interpolation)

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Similar to sinc() interpolation

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Connections with global polynomial interpolation and Vandermonde matrix
2. Roots of 1 on the Complex Plane

\[ \sqrt[3]{1} = \pm 1, \ \text{focus on the less obvious one: } -1. \]

\[ 3 \sqrt[3]{1} = 1, \ e^{j \frac{2\pi}{3}}, \ e^{-j \frac{2\pi}{3}} \]

Focus on the one with the angle:

\[ \omega_3 = e^{j \frac{2\pi}{3}} \]

\[ 3 \omega_3 = 1 \]

\[ N \sqrt[1]{1} = \text{small} \quad \text{"omega } N\text{"} \]

\[ \omega_N = e^{j \frac{2\pi}{N}} \]

\[ \omega_N = 1 \]

3. The DFT Matrix

Convention: Raw/Col indices go from 0, ..., N-1

DFT Matrix \( F_N \): \( (i, j) \text{th entry} = \omega_N^{ij} \)

Example: \( N=3 \)

\[
F_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \omega_3^0 & 1 \omega_3^0 & 1 \omega_3^0 \\
1 \omega_3^0 & 1 \omega_3^0 & 1 \omega_3^0 \\
1 \omega_3^0 & 1 \omega_3^0 & 1 \omega_3^0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 \\
1 & \omega_3 & \omega_3^2 \\
1 & \omega_3^2 & \omega_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 \\
1 & \omega_3 & \omega_3^2 \\
1 & \omega_3^2 & \omega_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 \\
1 & \omega_3 & \omega_3^2 \\
1 & \omega_3^2 & \omega_3
\end{bmatrix}
\]
DFT - VECTOR PRODUCT

Say \( \hat{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \) and \( \hat{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \) \( \in \mathbb{F}_{N} \hat{x} \)

Then: \( y_i = \sum_{j=0}^{N-1} \omega^{ij} x_j \), \( i = 0, \ldots, N-1 \) (WILL BE USEFUL LATER)

4. PROPERTIES OF THE DFT

A. SYMMETRIC: \( F_N^T = F_N \) (\( (i,j) \)th entry is \( \omega^{ij} \))

B. FULL RANK \iff\ INVERTIBLE

\[ F_N^{-1} = \frac{1}{N} F_N^* \] \( \iff \) INVERSE DFT = IDFT

\[ \text{OR: } F_N F_N^* = N I_N \] (ALMOST UNITARY)

PROOF:

Let \( A = F_N F_N^* \) \( \iff \) WILL SHOW THIS IS \( N I_N \)

What is \( a_{ij} \), the \((i,j)\)th entry of \( A \)?

\[ a_{ij} = \left( \text{i}\text{th row of } F_N^* \right) \times \left( \text{j}\text{th col of } F_N \right) \]

\[ = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{ik} \omega^{-jk} = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{(i-j)k} = \left[ \begin{array}{c} \omega^0 \\ \omega^1 \\ \vdots \\ \omega^{N-1} \end{array} \right] + \left[ \begin{array}{c} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^{N-2} \end{array} \right] + \cdots + \left[ \begin{array}{c} \omega^{i-1} \\ \omega^i \\ \vdots \\ \omega^{N-i} \end{array} \right] \]

\[ = \omega^{i-j} + \omega^{i-j} + \cdots + \omega^{i-j} \]

IF \( i = j \), then \( \omega^{i-j} = \omega^0 = 1 \)

\[ \Rightarrow a_{ii} = N \]

IF \( i \neq j \):

\[ a_{ij} = \omega^{i-j} + \omega^{i-j} + \cdots + \omega^{i-j} \]

\[ \Rightarrow a_{ij} - \delta_{ij} a_{ij} = \omega^{i-j} + \cdots + \omega^{i-j} \]

\[ \Rightarrow a_{ij} = \frac{1}{N} \left( \omega^{i-j} + \cdots + \omega^{i-j} \right) \]

\[ \Rightarrow a_{ij} = \frac{1}{N} \omega^{N(i-j)} = \frac{1}{N} \omega^{N(i-j)} \]

\[ \Rightarrow a_{ij} = \frac{1}{N} \omega^{i-j} \]

\[ \Rightarrow a_{ij} = \frac{1}{N} \omega^{i-j} \]

SUMMARY: \( a_{ij} = \begin{cases} \frac{1}{N} & i=j \\ 0 & \text{otherwise} \end{cases} \) \( \Rightarrow A = F_N F_N^* = N I_N \)
Q: Given that \( N \mathbf{n} = F_n F_n^* \), what property do the rows/cols of \( F_n \) have?

A: They are orthogonal (but not orthonormal)

Also: The sum of every row/col (except the first) = 0 (Sum of first row/col = \( N \)).

\[ \text{IDFT} = \text{DFT}^{-1} = \frac{1}{N} F_n^* \]

\[ \text{IDFT vector product} \]

\[ \mathbf{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \]

\[ = \text{IDFT} \cdot \mathbf{x} = \frac{1}{N} F_n^* \mathbf{x} \]

Then:

\[ y_i = \frac{1}{N} \sum_{j=0}^{N-1} w_i^j x_j, \quad i = 0, \ldots, N-1 \] (will be useful later)

5. Vectors from sequences

\[ \text{Turning a chunk of a sequence into a vector} \]

\[ \text{Given some discrete-time sequence:} \quad x[n], \quad n = -\infty, \ldots, -1, 0, 1, 2, \ldots \infty \]

\[ \text{Real valued} \]

\[ x[n] \]

\[ \cdots \]

\[ \cdots \]

\[ \cdots \]

\[ [n] \rightarrow \]

\[ \rightarrow \text{Pick any } N \text{ consecutive values} \]

\[ \rightarrow \text{Example:} \quad N = 5, \quad n = -5, -4, \ldots, -2, -1. \]

\[ x[n] \]

\[ \cdots \]

\[ \cdots \]

\[ \cdots \]

\[ [n] \rightarrow \]

\[ \text{N consecutive} \]

\[ \rightarrow \text{Size-N} \]

\[ \rightarrow \text{Make a vector of them:} \]

\[ \mathbf{x} = \begin{bmatrix} x[-5] \\ x[-4] \\ x[-3] \\ \vdots \\ x[-1] \end{bmatrix} \]
We can now run DFTs (or other vector operations) on \( \hat{x} \).

This is often useful.

E.g., the spectrogram shown earlier (demo/slides) does just this, using consecutive chunks of the input every time the display is updated.

Unless otherwise specified, we will assume below that vectors contain chunks from sequences, starting from \( t = 0 \).

\[
\hat{\mathbf{u}} = \begin{bmatrix}
    u[0] \\
    u[1] \\
    \vdots \\
    u[N-1]
\end{bmatrix}
\]

6. DFTs of Real Vectors: Complex Conjugacy Properties

Given a sequence chunk/vector \( \hat{\mathbf{z}} \) of length \( n \), all real numbers

i.e., \( \hat{\mathbf{z}} \in \mathbb{R}^n \)

Take its DFT:

\[
\hat{\mathbf{X}} = F_n \hat{\mathbf{z}}
\]

Then the members of \( \hat{\mathbf{X}} \) have a complex conjugacy property:

\[
\hat{X}_{n-1} = \overline{\hat{X}_1}, \quad (\imath = 1, \ldots, N-1)
\]

Note: Starting from 1 (\( \mathbf{u}[0] \)).

And: \( \hat{X}_0 \) is real.

In pictures:
IF \( N \) IS EVEN, i.e., \( N=2M \) FOR SOME NATURAL NUMBER \( M \)

THEN \( N-M = 2M-M = M \)

\[ \Rightarrow \overline{X_{M-i}} = X_i \]  
(with \( i=M \))  
\[ \overline{X_M} = X_M, \text{ i.e., } X_M \text{ is REAL.} \]

NOT TRUE IF \( N \) IS ODD

PROOF OF COMPLEX CONJUGACY PROPERTY

A) \( X_0 = \sum_{j=0}^{N-1} w_n x_i = \sum_{j=0}^{M-1} w_n x_i \). THIS IS REAL SINCE \( x_i \) ARE ALL REAL.

B) \( X_{M-i} = \sum_{j=0}^{N-1} w_n x_i = \sum_{j=0}^{M-1} w_n (-i) j x_i = \sum_{j=0}^{M-1} (\overline{w_n}) \overline{X_i} \)

\[ \overline{X_i} = \overline{x_i} \text{ BECAUSE } x_i \text{ IS REAL} \]

COROLLARY: SUPPOSE \( X_{M-i} = \overline{X_i} \), THEN \( \overline{X} \) IS NOT THE DFT OF A REAL VECTOR.

A SMALL EXERCISE FOR YOU

GIVEN SOME \( \overline{X} \in \mathbb{C}^N \) COMPLEX NUMBERS

OBEYS \( X_0 \) IS REAL AND \( X_{M-i} = \overline{X_i} \) FOR \( i=1,\ldots,N-1 \)

DEFINE \( \overline{z} = \text{IDFT} \in_T \overline{X} = \frac{1}{N} \mathcal{F}_N^* \overline{X} \)

PROVE THAT \( \overline{z} \) IS REAL.

FREQUENCY INTERPRETATION OF THE DFT

OUR GOAL HERE IS TO TRY TO DEVELOP A FEELING/INTUITION FOR WHAT THE ENTRIES OF \( \overline{X} \) REPRESENT.

WE WILL ASSUME \( \overline{z} \) (AND \( X(\ell) \)) ARE ALWAYS REAL.

THE FLOW IS LIKE THIS:

WE WILL MAKE UP SOME SIMPLE EXAMPLES OF \( \overline{X} \) (CONSISTENT WITH THE CONJUGACY CONDITION)

WE WILL IDFT \( \overline{X} \) TO GET \( \overline{z} \)

LOOKING AT THE ENTRIES OF \( \overline{z} \) WILL GIVE US INSIGHT.

\[ \overline{z} \]

\[ \overline{X}[\ell] \text{ CONSTANT WITH } \ell \in \{0,\ldots,N-1\} \]

\[ \overline{X}[0] \] DC
Choose $X_0 = \frac{a}{N}$, $X_f = 0$, $f = 1, 2, \ldots, N-1$

$X_f$ constant with $t = 0, 1, 2, \ldots, N-1$

Therefore, $X_0$ captures a DC waveform (interpreting $X_f$ as $X[0]$)

What about $X_1$?

Choose: $X_0 = 0$, $X_1 = a$, $X_{N-1} = \overline{a}$, the rest $= 0$

$x_f = \frac{1}{N} \sum_{t=0}^{N-1} w_n X_f = \frac{1}{N} \left( \sum_{t} w_n a + w_n \overline{a} \right) = \frac{1}{N} \left( \sum_{t} w_n a + w_n \overline{a} \right)$

$= \frac{2}{N} \text{Re} \left( \sum_{t} w_n a \right)$

say $a=1$ (for simplicity), then $x_f = \frac{2}{N} \cos \left( \frac{2\pi t}{N} \right)$

This is a (co)sine wave (discrete samples of it)
INTUITION SO FAR:

\[ X_0 \text{ gives you DC } x(t) \]

\[ X_1 \text{ (with complex conj term } X_{N} = X_1) \text{ gives you a sinusoidal } x(t) \text{ with period } N. \]